Eur. Phys. J. C 52, 721–733 (2007)
DOI 10.1140/epjc/s10052-007-0409-1

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Electromagnetic Casimir densities for a wedge with a coaxial cylindrical shell

A.A. Saharian[a]

Department of Physics, Yerevan State University, 1 Alex Manoogian Street, 375025 Yerevan, Armenia

Received: 24 July 2007 / Revised version: 25 August 2007 /
Published online: 15 September 2007 – © Springer-Verlag / Società Italiana di Fisica 2007

Abstract. Vacuum expectation values of the field square and the energy-momentum tensor for the electromagnetic field are investigated for the geometry of a wedge with a coaxial cylindrical boundary. All boundaries are assumed to be perfectly conducting, and both regions inside and outside the shell are considered. By using the generalized Abel–Plana formula, the vacuum expectation values are presented in the form of the sum of two terms. The first one corresponds to the geometry of the wedge without the cylindrical shell, and the second term is induced by the presence of the shell. The vacuum energy density induced by the shell is negative for the interior region and positive for the exterior region. The asymptotic behavior of the vacuum expectation values are investigated in various limiting cases. It is shown that the vacuum forces acting on the wedge sides due to the presence of the cylindrical boundary are always attractive.

PACS. 03.70.+k

1 Introduction

The Casimir effect is among the most interesting macroscopic manifestations of quantum fluctuations. It has important implications on all scales, from cosmological to subnuclear, and it has become in recent decades an increasingly popular topic in quantum field theory. In addition to its fundamental interest, the Casimir effect also plays an important role in the fabrication and operation of nano- and micro-scale mechanical systems. The imposition of boundary conditions on a quantum field leads to the modification of the spectrum for the zero-point fluctuations and results in the shift in the vacuum expectation values for physical quantities such as the energy density and stresses. In particular, the confinement of quantum fluctuations causes forces that act on constraining boundaries. The particular features of the resulting vacuum forces depend on the nature of the quantum field, the type of spacetime manifold, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir [1], much theoretical and experimental work has been done on this problem (see, e.g., [2–5] and references therein). Many different approaches have been used: the mode-summation method with combination of the zeta function regularization technique, the Green function formalism, multiple scattering expansions, heat-kernel series, etc. Advanced field-theoretical methods have been developed for Casimir calculations during the past years [6–22]. However, there are still difficulties in both interpretation and renormalization of the Casimir effect. Straightforward computations of the geometry dependencies are conceptually complicated, since relevant information is subtly encoded in the fluctuation spectrum [15–22]. Analytic solutions can usually be found only for highly symmetric geometries including planar, spherically and cylindrically symmetric boundaries. Recently the Casimir energy has been evaluated exactly for several less symmetric configurations of experimental interest. These include a sphere in front of a plane and a cylinder in front of a plane [23–26].

Investigations of quantum effects for cylindrical boundaries received a great deal of attention. In addition to the traditional problems of quantum electrodynamics in the presence of material boundaries, the Casimir effect for cylindrical geometries can also be important for flux tube models of confinement [27–29] and for determining the structure of the vacuum state in interacting field theories [30]. The calculation of the vacuum energy of electromagnetic field with boundary conditions defined on a cylinder turned out to be technically a more involved problem than the analogous one for a sphere. First, the Casimir energy of an infinite perfectly conducting cylindrical shell has been calculated in [31] by introducing an ultraviolet cutoff, and later the corresponding result was derived by the zeta function technique [32–34] (for a recent discussion of the Casimir energy and self-stresses in the more general case of a dielectric-diamagnetic cylinder, see [35–37] and references therein). The local characteristics of the corresponding electromagnetic vacuum, such as energy density and vacuum stresses, are considered in [38] for the

[a] e-mail: saharian@ictp.it

interior and exterior regions of a conducting cylindrical shell, and in [39] for the region between two coaxial shells (see also [40]). The vacuum forces acting on the boundaries in the geometry of two cylinders are also considered in [41–43]. The scalar Casimir densities for a single and two coaxial cylindrical shells with Robin boundary conditions are investigated in [44, 45]. The less symmetric configuration of two eccentric perfectly conducting cylinders is considered in [46]. The vacuum energy for a perfectly conducting cylinder of an elliptical section is evaluated in [47] by the mode-summation method, using the ellipticity as a perturbation parameter. The Casimir forces acting on two parallel plates inside a conducting cylindrical shell are investigated in [48].

Aside from their own theoretical and experimental interest, exactly solvable problems with this type of boundaries are useful for testing the validity of various approximations used to deal with more complicated geometries. From this point of view the wedge with a coaxial cylindrical boundary is an interesting system, since the geometry is non-trivial and it includes two dynamical parameters, the radius of the cylindrical shell and the opening angle of the wedge. This geometry is also interesting from the point of view of a general analysis of surface divergences in the expectation values of local physical observables for boundaries with discontinuities. The non-smoothness of the boundary generates additional contributions to the heat-kernel coefficients (see, for instance, the discussion in [49–51] and references therein). The present paper is concerned with a local analysis of the vacuum of the electromagnetic field constrained to satisfy perfectly conducting boundary conditions on the boundary surfaces of a wedge with a coaxial cylindrical boundary. Namely, we will study the vacuum expectation values of the field squares and the energy-momentum tensor for the electromagnetic field for both regions inside and outside the cylindrical shell. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modeling the self-consistent dynamics involving the gravitational field. The vacuum expectation value of the square of the electric field determines the electromagnetic force on a neutral polarizable particle. Some investigations most relevant to the present paper are contained in [2, 52–57], where the geometry of a wedge without a cylindrical boundary is considered for a conformally coupled scalar and electromagnetic fields in a four dimensional spacetime. The total Casimir energy of a semi-circular infinite cylindrical shell with perfectly conducting walls is considered in [58] by using the zeta function technique. For a scalar field with an arbitrary curvature coupling parameter, the Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor in the geometry of a wedge with an arbitrary opening angle and with a cylindrical boundary are evaluated in [59, 60]. Note that, unlike the case of conformally coupled fields, for general coupling the vacuum energy-momentum tensor is angle-dependent and diverges on the wedge sides. Our method here employs the mode summation and is based on a variant of generalized Abel–Plana formulae [40, 61, 62]. This enables us to extract from the vacuum expectation values the parts due to a wedge without the cylindrical shell and to present the parts induced by the shell in terms of strongly convergent integrals. Note that the closely related problem of the vacuum densities induced by a cylindrical boundary in the geometry of a cosmic string is investigated in [63, 64] for both scalar and electromagnetic fields.

We have organized the paper as follows. In the next section we describe the structure of the modes for a wedge with a cylindrical shell in the region inside the shell. By applying to the corresponding mode sums the generalized Abel–Plana formula, we evaluate the vacuum expectation values of the electric and magnetic field square. Various limiting cases of the general formulae are discussed. Section 3 is devoted to the investigation of the vacuum expectation values for the energy-momentum tensor of the electromagnetic field in the region inside the shell. The additional vacuum forces acting on the wedge sides due to the presence of the cylindrical boundary are evaluated. In Sect. 4 we consider the vacuum densities for a wedge with the cylindrical shell in the exterior region with respect to the shell. Formulae for the shell contributions are derived and the corresponding surface divergences are investigated. The vacuum forces acting on the wedge sides are discussed. The main results are summarized and discussed in Sect. 5.

2 Vacuum expectation values of the field square inside a cylindrical shell

Consider a wedge with the opening angle ϕ_0 and with a coaxial cylindrical boundary of radius a (see Fig. 1), assuming that all boundaries are perfectly conducting. In accordance with the problem's symmetry, in the discussion below cylindrical coordinates (r, ϕ, z) will be used. We are interested in the vacuum expectation values (VEVs) of the field square and the energy-momentum tensor for the electromagnetic field. Expanding the field operator in terms of creation and annihilation operators and using the commutation relations, the VEV for a quantity $F\{A_i, A_k\}$ bilinear in the field can be presented in the form of the mode sum:

$$\langle 0|F\{A_i, A_k\}|0\rangle = \sum_{\alpha} F\{A_{\alpha i}, A^*_{\alpha k}\}, \tag{1}$$

where $\{A_{\alpha i}, A^*_{\alpha k}\}$ is a complete set of solutions of classical field equations satisfying the boundary conditions on the bounding surfaces and specified by a set of quantum numbers α.

In accordance with (1), for the evaluation of the VEVs for the square of the electric and magnetic fields and the energy-momentum tensor, the corresponding eigenfunctions are needed. In this section we consider the region inside the cylindrical shell (region I in Fig. 1). For the geometry under consideration, there are two different types of eigenfunctions, corresponding to transverse magnetic (TM) and transverse electric (TE) waves. In the discus-

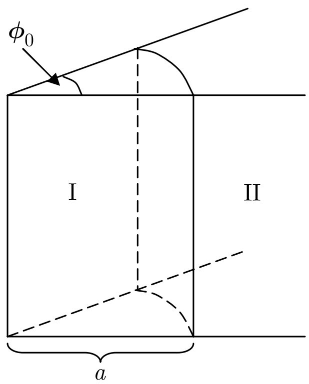

Fig. 1. Geometry of a wedge with a coaxial cylindrical boundary with radius a

sion below we will specify these modes by the index $\lambda=0$ and $\lambda=1$ for the TM and TE waves, respectively. In the Coulomb gauge, the vector potentials for the TM and TE modes are given by the formulae (here and below Gaussian units are used)

$$\mathbf{A}_\alpha=\beta_\alpha\begin{cases}(1/\mathrm{i}\omega)\left(\gamma^2\mathbf{e}_3+\mathrm{i}k\nabla_\mathrm{t}\right)J_{q|m|}(\gamma r)\sin(qm\phi) & \\ \quad\times\exp\left[\mathrm{i}\left(kz-\omega t\right)\right], & \lambda=0\,,\\ -\mathbf{e}_3\times\nabla_\mathrm{t}\left\{J_{q|m|}(\gamma r)\cos(qm\phi)\right. & \\ \quad\left.\times\exp\left[\mathrm{i}\left(kz-\omega t\right)\right]\right\}, & \lambda=1\,,\end{cases} \tag{2}$$

where $\mathbf{e}_3$ is the unit vector along the axis of the wedge, ∇_t is the part of the nabla operator transverse to this axis, $J_\nu(x)$ is a Bessel function of the first kind, and

$$\omega^2=\gamma^2+k^2,\quad q=\pi/\phi_0\,. \tag{3}$$

In (2), $m=1,2,\ldots$ for $\lambda=0$ and $m=0,1,2,\ldots$ for $\lambda=1$. Equations (2) for the fundamental TM and TE modes are obtained from the corresponding formulae for a cylindrical waveguide (see, for instance, [65]) by imposing the perfect conductor boundary conditions on the wedge sides $\phi=0$ and $\phi=\phi_0$. The normalization coefficient β_α is found from the orthonormalization condition for the vector potential:

$$\int\mathrm{d}V\,\mathbf{A}_\alpha\cdot\mathbf{A}^*_{\alpha'}=\frac{2\pi}{\omega}\delta_{\alpha\alpha'}\,, \tag{4}$$

where the integration goes over the region inside the shell. From this condition, by using the standard integral involving the square of the Bessel function, one finds

$$\beta^2_\alpha=\frac{4qT_{qm}(\gamma a)}{\pi\omega a\gamma}\delta_m,\quad \delta_m=\begin{cases}1/2, & m=0\,,\\ 1, & m\neq0\,,\end{cases} \tag{5}$$

where we have introduced the notation

$$T_\nu(x)=x\left[J'^2_\nu(x)+(1-\nu^2/x^2)J^2_\nu(x)\right]^{-1}. \tag{6}$$

The eigenfunctions (2) satisfy the standard boundary conditions for the electric and magnetic fields, $\mathbf{n}\times\mathbf{E}=0$ and $\mathbf{n}\cdot\mathbf{B}=0$, on the wedge sides corresponding to $\phi=0$ and $\phi=\phi_0$, with $\mathbf{n}$ being the normal to the boundary. The eigenvalues for the quantum number γ are determined by the boundary conditions on the cylindrical shell. From the latter it follows that these eigenvalues are solutions of

$$J^{(\lambda)}_{qm}(\gamma a)=0,\quad \lambda=0,1\,, \tag{7}$$

where we use the notation $J^{(0)}_\nu(x)=J_\nu(x)$ and $J^{(1)}_\nu(x)=J'_\nu(x)$. We will denote the corresponding eigenmodes by $\gamma a=j^{(\lambda)}_{m,n}$, $n=1,2,\ldots$, assuming that the zeros $j^{(\lambda)}_{m,n}$ are arranged in ascending order. Consequently, the eigenfunctions are specified by the set of quantum numbers $\alpha=(k,m,\lambda,n)$.

First we consider the VEVs of the squares of the electric and magnetic fields inside the shell. Substituting the eigenfunctions (2) into the corresponding mode-sum formula, we find

$$\langle0|F^2|0\rangle=\frac{4q}{\pi a^3}{\sum_{m=0}^{\infty}}'\int_{-\infty}^{+\infty}\mathrm{d}k\sum_{\lambda=0,1}\sum_{n=1}^{\infty}\frac{j^{(\lambda)3}_{m,n}T_{qm}\left(j^{(\lambda)}_{m,n}\right)}{\sqrt{j^{(\lambda)2}_{m,n}+k^2a^2}}\times g^{(\eta_{F\lambda})}\left[\Phi^{(\lambda)}_{qm}(\phi),J_{qm}(j^{(\lambda)}_{m,n}r/a)\right], \tag{8}$$

where $F=E,B$ with $\eta_{E\lambda}=\lambda$, $\eta_{B\lambda}=1-\lambda$, and the prime in the summation over m means that the term $m=0$ should be halved. In (8), for a given function $f(x)$, we have introduced the notation

$$g^{(0)}[\Phi(\phi),f(x)]=\left(k^2r^2/x^2\right)\left[\Phi^2(\phi)f'^2(x)+\Phi'^2(\phi)f^2(x)/x^2\right]+\Phi^2(\phi)f^2(x), \tag{9}$$

$$g^{(1)}[\Phi(\phi),f(x)]=\left(1+k^2r^2/x^2\right)\left[\Phi^2(\phi)f'^2(x)+\Phi'^2(\phi)f^2(x)/x^2\right], \tag{10}$$

and

$$\Phi^{(\lambda)}_\nu(\phi)=\begin{cases}\sin(\nu\phi), & \lambda=0\,,\\ \cos(\nu\phi), & \lambda=1\,.\end{cases} \tag{11}$$

Equations (8), corresponding to the electric and magnetic fields, are divergent. They may be regularized introducing a cutoff function $\psi_\mu(\omega)$ with the cutting parameter μ, which makes the divergent expressions finite and satisfies the condition $\psi_\mu(\omega)\to1$ for $\mu\to0$. After the renormalization the cutoff function is removed by taking the limit $\mu\to0$. An alternative way is to consider the product of the fields at different spacetime points and to take the coincidence limit after the subtraction of the corresponding Minkowskian part. Our approach here follows the first method.

As we do not know the explicit expressions for the zeros $j^{(\lambda)}_{m,n}$ as functions on m and n, and the summand in (8) is a strongly oscillating function for large values of m and n, this formula is not convenient for the further evaluation of the VEVs of the field square. In order to obtain an alternative representation, we apply to the series over n the generalized Abel–Plana summation formula [61, 62],

$$\sum_{n=1}^{\infty} T_{qm}\left(j_{m,n}^{(\lambda)}\right) f(j_{m,n}^{(\lambda)}) = \frac{1}{2}\int_0^{\infty} \mathrm{d}x\, f(x) + \frac{\pi}{4}\operatorname*{Res}_{z=0} f(z)\frac{Y_{qm}^{(\lambda)}(z)}{J_{qm}^{(\lambda)}(z)} - \frac{1}{2\pi}\int_0^{\infty}\mathrm{d}x\,\frac{K_{qm}^{(\lambda)}(x)}{I_{qm}^{(\lambda)}(x)} \times\left[\mathrm{e}^{-qm\pi\mathrm{i}} f(\mathrm{e}^{\pi\mathrm{i}/2}x) + \mathrm{e}^{qm\pi\mathrm{i}} f(\mathrm{e}^{-\pi\mathrm{i}/2}x)\right], \quad (12)$$

where $Y_\nu(z)$ is the Neumann function and $I_\nu(z)$, $K_\nu(z)$ are the modified Bessel functions. As can be seen, for points away from the shell the contribution to the VEVs coming from the second integral term on the right-hand side of (12) is finite in the limit $\mu \to 0$ and, hence, the cutoff function in this term can safely be removed. As a result the VEVs can be written in the form

$$\langle 0|F^2|0\rangle = \langle 0_{\mathrm{w}}|F^2|0_{\mathrm{w}}\rangle + \langle F^2\rangle_{\mathrm{cyl}}, \quad (13)$$

where

$$\langle 0_{\mathrm{w}}|F^2|0_{\mathrm{w}}\rangle = \frac{q}{\pi}\sum_{m=0}^{\infty}{}' \int_{-\infty}^{+\infty}\mathrm{d}k \int_0^{\infty}\mathrm{d}\gamma\,\frac{\gamma^3\psi_\mu(\omega)}{\sqrt{\gamma^2+k^2}} \times\left\{\left(1+\frac{2k^2}{\gamma^2}\right)\left[J_{qm}'^2(\gamma r) + \frac{q^2m^2}{\gamma^2 r^2}J_{qm}^2(\gamma r)\right] + J_{qm}^2(\gamma r) - (-1)^{\eta_{F1}}\cos(2qm\phi) \times\left[J_{qm}'^2(\gamma r) - \left(1+\frac{q^2m^2}{\gamma^2r^2}\right)J_{qm}^2(\gamma r)\right]\right\}, \quad (14)$$

and

$$\langle F^2\rangle_{\mathrm{cyl}} = \frac{8q}{\pi^2}\sum_{m=0}^{\infty}{}'\int_0^{\infty}\mathrm{d}k \sum_{\lambda=0,1}\int_k^{\infty}\mathrm{d}x\, x^3 \times\frac{K_{qm}^{(\lambda)}(xa)}{I_{qm}^{(\lambda)}(xa)}\frac{G^{(\eta_{F\lambda})}\left[k, \Phi_{qm}^{(\lambda)}(\phi), I_{qm}(xr)\right]}{\sqrt{x^2-k^2}}. \quad (15)$$

In (15) we have introduced the notation

$$G^{(0)}[k,\Phi(\phi), f(x)] = \left(k^2r^2/x^2\right)\left[\Phi^2(\phi)f'^2(x) + \Phi'^2(\phi)f^2(x)/x^2\right] + \Phi^2(\phi)f^2(x), \quad (16)$$

$$G^{(1)}[k,\Phi(\phi), f(x)] = \left(k^2r^2/x^2 - 1\right)\left[\Phi^2(\phi)f'^2(x) + \Phi'^2(\phi)f^2(x)/x^2\right]. \quad (17)$$

The second term on the right-hand side of (13) vanishes in the limit $a \to \infty$, and the first one does not depend on a. Thus, we can conclude that the term $\langle 0_{\mathrm{w}}|F^2|0_{\mathrm{w}}\rangle$ corresponds to the part in the VEVs when the cylindrical shell is absent with the corresponding vacuum state $|0_{\mathrm{w}}\rangle$. Hence, the application of the generalized Abel–Plana formula enables us to extract from the VEVs the parts induced by the cylindrical shell without specifying the cutoff function. In addition, these parts are presented in terms of the exponentially convergent integrals.

First, let us concentrate on the part corresponding to the wedge without a cylindrical shell. First of all we note that in (14) the part that does not depend on the angular coordinate ϕ is the same as in the corresponding problem of the cosmic string geometry with the angle deficit $2\pi - \phi_0$ [64], which we shall denote by $\langle 0_{\mathrm{s}}|F^2|0_{\mathrm{s}}\rangle$. For this part we have

$$\langle 0_{\mathrm{s}}|F^2|0_{\mathrm{s}}\rangle = \frac{q}{\pi}\sum_{m=0}^{\infty}{}'\int_{-\infty}^{+\infty}\mathrm{d}k\int_0^{\infty}\mathrm{d}\gamma\,\frac{\gamma^3\psi_\mu(\omega)}{\sqrt{\gamma^2+k^2}} \times\left\{\left(1+\frac{2k^2}{\gamma^2}\right)\left[J_{qm}'^2(\gamma r) + \frac{q^2m^2}{\gamma^2r^2}J_{qm}^2(\gamma r)\right] + J_{qm}^2(\gamma r)\right\} = \langle 0_{\mathrm{M}}|F^2|0_{\mathrm{M}}\rangle - \frac{(q^2-1)(q^2+11)}{180\pi r^4}, \quad (18)$$

where $\langle 0_{\mathrm{M}}|F^2|0_{\mathrm{M}}\rangle$ is the part corresponding to the Minkowskian spacetime without boundaries and in the last expression we have removed the cutoff. To evaluate the part in (14) that depends on ϕ, we firstly consider the case when the parameter q is an integer. In this case the summation over m can be done by using [66, 67]

$$\sum_{m=0}^{\infty}{}'\cos(2qm\phi)J_{qm}^2(y) = \frac{1}{2q}\sum_{l=0}^{q-1}J_0(2y\sin(\phi+\phi_0 l)). \quad (19)$$

The formulae for the other series entering in (14) are obtained from (19) taking the derivatives with respect to ϕ and y. In particular, for the combination appearing in the angle-dependent part, we obtain

$$\sum_{m=0}^{\infty}{}'\cos(2qm\phi)\left[J_{qm}'^2(y) - \left(1+\frac{q^2m^2}{y^2}\right)J_{qm}^2(y)\right] = -\frac{1}{q}\sum_{l=0}^{q-1}J_1'\left(2y\sin\left(\phi+\phi_0 l\right)\right). \quad (20)$$

Substituting this in (14), the remaining integrals are evaluated by introducing polar coordinates in the (k,γ)-plane. In this way one finds

$$\langle 0_{\mathrm{w}}|F^2|0_{\mathrm{w}}\rangle = \langle 0_{\mathrm{s}}|F^2|0_{\mathrm{s}}\rangle - \frac{3(-1)^{\eta_{F1}}}{4\pi r^4}\sum_{l=0}^{q-1}\sin^{-4}(\phi + l\pi/q). \quad (21)$$

The sum on the right-hand side of this formula is evaluated by the double differentiation of the relation [66]

$$\sum_{l=0}^{q-1}\cos^{-2}(x + l\pi/q) = q^2\sin^{-2}(qx + q\pi/2). \quad (22)$$

Finally, for the renormalized VEVs of the field square in the geometry of a wedge without a cylindrical boundary we find

$$\langle F^2\rangle_{\rm w,ren} = -\frac{(q^2-1)(q^2+11)}{180\pi r^4} - \frac{(-1)^{\eta_{F1}} q^2}{2\pi r^4 \sin^2(q\phi)}\left[\frac{3q^2}{2\sin^2(q\phi)} + 1 - q^2\right], \quad (23)$$

with $\eta_{E1} = 1$ and $\eta_{B1} = 0$. Though we have derived this formula for integer values of the parameter q, by the analytic continuation it is valid for non-integer values of this parameter as well. The expression on the right of (23) is invariant under the replacement $\phi \to \phi_0 - \phi$, and, as we could expect, the VEVs are symmetric with respect to the half-plane $\phi = \phi_0/2$. Equation (23) for $F = E$ was derived in [55] within the framework of Schwinger's source theory. For $q = 1$ from (23) as a special case we obtain the renormalized VEVs of the field square for a conducting plate. In this case $x = r\sin\phi$ is the distance from the plate and one has

$$\langle F^2\rangle_{\rm pl,ren} = -\frac{3(-1)^{\eta_{F1}}}{4\pi x^4}. \quad (24)$$

Another special case, $q = 1/2$, corresponds to the geometry of a half-plane. In (23) taking the limit $r \to \infty$, with $x_0 = r\phi_0$ being fixed, we obtain the corresponding results in the region between two parallel plates located at the points $x = 0$ and $x = x_0$:

$$\langle F^2\rangle_{\rm 2pl,ren} = -\frac{\pi^3}{180 x_0^4} - \frac{(-1)^{\eta_{F1}}\pi^3}{2x_0^4 \sin^2(\pi x/x_0)} \times \left[\frac{3}{2\sin^2(\pi x/x_0)} - 1\right]. \quad (25)$$

Now, we turn to the investigation of the parts in the VEVs of the field square induced by the cylindrical boundary and given by (15). By using

$$\int_0^\infty {\rm d}k\, k^m \int_k^\infty {\rm d}x\, \frac{x f(x)}{\sqrt{x^2-k^2}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{m}{2}+1\right)} \int_0^\infty {\rm d}x\, x^{m+1} f(x), \quad (26)$$

these parts are presented in the form

$$\langle F^2\rangle_{\rm cyl} = \frac{2q}{\pi} \sum_{m=0}^{\infty}{}' \sum_{\lambda=0,1} \int_0^\infty {\rm d}x\, x^3 \frac{K_{qm}^{(\lambda)}(xa)}{I_{qm}^{(\lambda)}(xa)} G^{(\eta_{F\lambda})} \times \left[\Phi_{qm}^{(\lambda)}(\phi), I_{qm}(xr)\right]. \quad (27)$$

Here, for given functions $f(x)$ and $\Phi(\phi)$, we have introduced the notation

$$G^{(0)}[\Phi(\phi), f(x)] = \Phi^2(\phi) f'^2(x) + \Phi'^2(\phi) f^2(x)/x^2 + 2\Phi^2(\phi) f^2(x), \quad (28)$$

$$G^{(1)}[\Phi(\phi), f(x)] = -\Phi^2(\phi) f'^2(x) - \Phi'^2(\phi) f^2(x)/x^2. \quad (29)$$

As we can see, the parts in the VEVs induced by the cylindrical shell are symmetric with respect to the half-plane $\phi = \phi_0/2$.

The expression in the right-hand side of (27) is finite for $0 < r < a$ including the points on the wedge sides, and it diverges on the shell. To find the leading term in the corresponding asymptotic expansion, we note that near the shell the main contribution comes from large values of m. By using the uniform asymptotic expansions of the modified Bessel functions (see, for instance, [68]) for large values of the order, up to the leading order, for the points $a - r \ll a|\sin\phi|, a|\sin(\phi_0 - \phi)|$ we find

$$\langle F^2\rangle_{\rm cyl} \approx -\frac{3(-1)^{\eta_{F1}}}{4\pi(a-r)^4}. \quad (30)$$

For the points near the edges ($r = a, \phi = 0, \phi_0$) the leading terms in the corresponding asymptotic expansions are the same as for the geometry of a wedge with the opening angle $\phi_0 = \pi/2$. The leading terms given by (30) are the same as for the geometry of a single plate (see (24)). They do not depend on ϕ_0 and have opposite signs for the electric and magnetic fields. In particular, the leading terms are cancelled in the evaluation of the vacuum energy density. Surface divergences originate in the unphysical nature of the perfect conductor boundary conditions and are well-known in quantum field theory with boundaries. In reality the expectation values will attain a limiting value on the conductor surface that will depend on the molecular details of the conductor. From the formulae given above it follows that the main contributions to $\langle F^2\rangle_{\rm cyl}$ are due to the frequencies $\omega \lesssim (a-r)^{-1}$. Hence, we expect that (27) is valid for real conductors up to distances r for which $(a-r)^{-1} \ll \omega_0$, with ω_0 being the characteristic frequency, such that for $\omega > \omega_0$ the conditions for perfect conductivity fail.

Near the edge $r = 0$, assuming that $r/a \ll 1$, the asymptotic behavior of the part induced in the VEVs of the field square by the cylindrical shell depends on the parameter q. For $q > 1 + \eta_{F1}$, the dominant contribution comes from the lowest mode $m = 0$ and to the leading order one has

$$\langle F^2\rangle_{\rm cyl} \approx -(-1)^{\eta_{F1}} \frac{2^{1-\eta_{F1}} q}{\pi a^4} \left(\frac{r}{2a}\right)^{2\eta_{F1}} \int_0^\infty {\rm d}x\, x^3 \frac{K_1(x)}{I_1(x)}. \quad (31)$$

In this case the quantity $\langle B^2\rangle_{\rm cyl}$ takes a finite limiting value on the edge $r = 0$, whereas $\langle E^2\rangle_{\rm cyl}$ vanishes as r^2. For $q < 1 + \eta_{F1}$ the main contribution comes from the mode with $m = 1$ and the shell-induced parts diverge on the edge $r = 0$. The leading terms are given by

$$\langle F^2\rangle_{\rm cyl} \approx -\frac{(-1)^{\eta_{F1}} q (r/a)^{2(q-1)}}{2^{2q-1}\pi\Gamma^2(q) a^4} \times \int_0^\infty {\rm d}x\, x^{2q+1} \left[\frac{K_q(x)}{I_q(x)} - \frac{K'_q(x)}{I'_q(x)}\right]. \quad (32)$$

As for the points near the shell, here the leading divergences in the VEVs of the electric and magnetic fields are cancelled in the evaluation of the vacuum energy density. For $q = 1 + \eta_{F1}$ the main contribution comes from the modes $m=0,1$ and the corresponding asymptotic behavior is obtained by summing the right-hand sides of (31) and (32). In accordance with (23), near the edge, $r=0$, the total VEV is dominated by the part coming from the wedge without the cylindrical shell. Note that the quantity $a^4\langle F^2\rangle_{\rm cyl}$ depends on the radial coordinate through the ratio r/a and, hence, (31) and (32) also describe the asymptotic behavior of the VEVs for large values of the cylindrical shell radius when r is fixed. In this limit the VEVs vanish as $1/a^{4+2\eta_{F1}}$ for $q>1+\eta_{F1}$, and like $1/a^{2+2q}$ for $q<1+\eta_{F1}$. Here we have considered the VEVs for the field square. The VEVs for the bilinear products of the fields at different spacetime points may be evaluated in a similar way.

Now, we turn to the investigation of the behavior of the VEVs induced by the cylindrical boundary in the limit $q\gg 1$. In this limit the order of the modified Bessel functions is large for $m\neq 0$. By using the corresponding asymptotic formulae, it can be seen that the contribution of these terms is suppressed by the factor $\exp[-2qm\ln(a/r)]$. As a result, the main contribution comes from the lowest mode, $m=0$, and the VEVs induced by the cylindrical shell are proportional to q. Note that in this limit the part corresponding to the wedge without the cylindrical shell behaves as q^4.

3 Vacuum energy-momentum tensor inside the cylindrical shell

Now let us consider the VEV of the energy-momentum tensor in the region inside the cylindrical shell. Substituting the eigenfunctions (2) into the corresponding mode-sum formula, for the non-zero components we obtain (no summation over i)

$$\begin{aligned}\langle 0|T_i^i|0\rangle &= \frac{q}{2\pi^2 a^3}\sideset{}{'}\sum_{m=0}^{\infty}\int_{-\infty}^{+\infty}\mathrm{d}k\sum_{\lambda=0,1}\sum_{n=1}^{\infty}\frac{j_{m,n}^{(\lambda)3}T_{qm}\left(j_{m,n}^{(\lambda)}\right)}{\sqrt{j_{m,n}^{(\lambda)2}+k^2a^2}}\\ &\quad\times f^{(i)}\left[\Phi_{qm}^{(\lambda)}(\phi), J_{qm}(j_{m,n}^{(\lambda)}r/a)\right], \qquad (33)\\ \langle 0|T_2^1|0\rangle &= -\frac{q^2}{4\pi^2 a}\frac{\partial}{\partial r}\sideset{}{'}\sum_{m=0}^{\infty} m\sin(2qm\phi)\int_{-\infty}^{+\infty}\mathrm{d}k\sum_{\lambda=0,1}(-1)^{\lambda}\\ &\quad\times\sum_{n=1}^{\infty}\frac{j_{m,n}^{(\lambda)}T_{qm}\left(j_{m,n}^{(\lambda)}\right)}{\sqrt{j_{m,n}^{(\lambda)2}+k^2a^2}}J_{qm}^2\left(j_{m,n}^{(\lambda)}r/a\right), \qquad (34)\end{aligned}$$

where $i=0,1,2,3$, and we have introduced the notation

$$\begin{aligned}f^{(j)}[\Phi(\phi),f(x)] &= (-1)^j\left(2k^2/\gamma^2+1\right)\\ &\quad\times\left[\Phi^2(\phi)f'^2(x)+\Phi'^2(\phi)f^2(x)/y^2\right]\\ &\quad+\Phi^2(\phi)f^2(x), \qquad (35)\\ f^{(l)}[\Phi(\phi),f(x)] &= (-1)^l\Phi^2(\phi)f'^2(x)\\ &\quad-\left[\Phi^2(\phi)+(-1)^l\Phi'^2(\phi)/x^2\right]f^2(x), \qquad (36)\end{aligned}$$

with $j=0,3$ and $l=1,2$. As in the case of the field square, in (33) and (34) we introduce a cutoff function and apply (12) for the summation over n. This enables us to present the vacuum energy-momentum tensor in the form of the sum

$$\langle 0|T_i^k|0\rangle = \langle 0_{\rm w}|T_i^k|0_{\rm w}\rangle + \langle T_i^k\rangle_{\rm cyl}, \qquad (37)$$

where $\langle 0_{\rm w}|T_i^k|0_{\rm w}\rangle$ is the part corresponding to the geometry of a wedge without a cylindrical boundary and $\langle T_i^k\rangle_{\rm cyl}$ is induced by the cylindrical shell. By taking into account (26), the latter may be written in the form (no summation over i)

$$\begin{aligned}\langle T_i^i\rangle_{\rm cyl} &= \frac{q}{2\pi^2}\sideset{}{'}\sum_{m=0}^{\infty}\sum_{\lambda=0,1}\int_0^{\infty}\mathrm{d}x x^3\frac{K_{qm}^{(\lambda)}(xa)}{I_{qm}^{(\lambda)}(xa)}\\ &\quad\times F^{(i)}\left[\Phi_{qm}^{(\lambda)}(\phi), I_{qm}(xr)\right], \qquad (38)\\ \langle T_2^1\rangle_{\rm cyl} &= \frac{q^2}{4\pi^2}\frac{\partial}{\partial r}\sideset{}{'}\sum_{m=0}^{\infty} m\sin(2qm\phi)\sum_{\lambda=0,1}(-1)^{\lambda}\\ &\quad\times\int_0^{\infty}\mathrm{d}x x\frac{K_{qm}^{(\lambda)}(xa)}{I_{qm}^{(\lambda)}(xa)}I_{qm}^2(xr), \qquad (39)\end{aligned}$$

with the notation

$$\begin{aligned}F^{(i)}[\Phi(\phi),f(y)] &= \Phi^2(\phi)f^2(y), \quad i=0,3, \qquad (40)\\ F^{(i)}[\Phi(\phi),f(y)] &= -(-1)^i\Phi^2(\phi)f'^2(y)\\ &\quad-\left[\Phi^2(\phi)-(-1)^i\Phi'^2(\phi)/y^2\right]f^2(y),\\ &\quad i=1,2. \qquad (41)\end{aligned}$$

The diagonal components are symmetric with respect to the half-plane $\phi=\phi_0/2$, whereas the off-diagonal component is an odd function under the replacement $\phi\to\phi_0-\phi$. As can easily be checked, the tensor $\langle T_i^k\rangle_{\rm cyl}$ is traceless and satisfies the covariant continuity equation $\nabla_k\langle T_i^k\rangle_{\rm cyl}=0$. For the geometry under consideration the latter leads to

$$\frac{\partial}{\partial r}\left(r\langle T_2^1\rangle_{\rm cyl}\right)+r\frac{\partial}{\partial\phi}\langle T_2^2\rangle_{\rm cyl}=0, \qquad (42)$$

$$\frac{\partial}{\partial r}\left(r\langle T_1^1\rangle_{\rm cyl}\right)+r\frac{\partial}{\partial\phi}\langle T_1^2\rangle_{\rm cyl}=\langle T_2^2\rangle_{\rm cyl}. \qquad (43)$$

As is seen from (39), the off-diagonal component $\langle T_2^1\rangle_{\rm cyl}$ vanishes at the wedge sides and for these points the VEV of the energy-momentum tensor is diagonal. By using the inequalities $I'_\nu(x)<\sqrt{1+\nu^2/x^2}I_\nu(x)$ and $-K'_\nu(x)>\sqrt{1+\nu^2/x^2}K_\nu(x)$ for the modified Bessel functions, it can be seen that $K'_\nu(x)/I'_\nu(x)+K_\nu(x)/I_\nu(x)<0$. From this relation it follows that the vacuum energy density induced by the cylindrical shell in the interior region is always negative.

The renormalized VEV of the energy-momentum tensor for the geometry without the cylindrical shell is obtained by using the corresponding formulae for the field square. For the corresponding energy density one finds

$$\langle T_0^0 \rangle_{\rm w,ren} = \frac{1}{8\pi}\left(\langle E^2 \rangle_{\rm w,ren} + \langle B^2 \rangle_{\rm w,ren}\right) = -\frac{(q^2-1)(q^2+11)}{720\pi^2 r^4}. \quad (44)$$

As we see, the parts in the VEVs of the field square that diverge on the wedge sides cancel out, and the corresponding energy density is finite everywhere except the edge. Equation (44) coincides with the corresponding result for the geometry of the cosmic string [69, 70] with the angle deficit $2\pi - \phi_0$ and in the corresponding formula $q = 2\pi/\phi_0$. Other components are found from the tracelessness condition and the continuity equation and one has [2, 52, 53]

$$\langle T_i^k \rangle_{\rm w,ren} = -\frac{(q^2-1)(q^2+11)}{720\pi^2 r^4}{\rm diag}(1,1,-3,1). \quad (45)$$

As we could expect this VEV vanishes for the geometry of a single plate corresponding to $q = 1$. In the limit $r \to \infty$, for fixed values $x_0 = r\phi_0$, from (45) the standard result for the geometry of two parallel conducting plates is obtained.

In general, due to the surface divergences in the VEVs of the local physical observables, the vacuum forces acting on the constraining boundaries cannot be directly evaluated by using the corresponding vacuum stresses. However, for the electromagnetic field the case of plane boundaries is an exception. The reason is that for a single perfectly conducting plane boundary the VEV of the energy-momentum tensor vanishes and for the geometry under consideration the non-trivial contributions come about by the presence of the second wedge and cylindrical shell. As a result, for the geometry of a wedge with cylindrical shell the VEV of the energy-momentum tensor is finite on the wedge sides (except the points on the edges $r = 0$ and $r = a$). The corresponding renormalization procedure is the same as in quantum field theory without boundaries, and no counterterms located on the boundary are needed. The normal force acting on the wedge sides is determined by the component $\langle T_2^2 \rangle_{\rm ren}$ of the vacuum energy-momentum tensor evaluated for $\phi = 0$ and $\phi = \phi_0$. On the basis of (37) for the corresponding effective pressure one has

$$p_2 = -\langle T_2^2 \rangle_{\rm ren}|_{\phi=0,\phi_0} = p_{\rm 2w} + p_{\rm 2cyl}, \quad (46)$$

where

$$p_{\rm 2w} = -\frac{(q^2-1)(q^2+11)}{240\pi^2 r^4} \quad (47)$$

is the normal force acting per unit surface of the wedge for the case without a cylindrical boundary, and the additional term

$$p_{\rm 2cyl} = -\langle T_2^2 \rangle_{\rm cyl}|_{\phi=0,\phi_0} = -\frac{q}{\pi^2}\sum_{m=0}^{\infty}{}' \sum_{\lambda=0,1}\int_0^\infty {\rm d}x x^3 \frac{K_{qm}^{(\lambda)}(xa)}{I_{qm}^{(\lambda)}(xa)} F_{qm}^{(\lambda)}\left[I_{qm}(xr)\right], \quad (48)$$

with the notation

$$F_\nu^{(\lambda)}[f(y)] = \begin{cases} \nu^2 f^2(y)/y^2, & \lambda = 0, \\ -f'^2(y) - f^2(y), & \lambda = 1 \end{cases} \quad (49)$$

is induced by the cylindrical shell. As has been mentioned the finiteness of the normal stress on the wedge sides is a consequence of the fact that for a single perfectly conducting plane boundary this stress vanishes. Note that this result can be directly obtained from the symmetry of the corresponding problem on combination with of the continuity equation for the energy-momentum tensor. It also survives for more realistic models of the plane boundary (see, for instance, [71–73]) though the corresponding energy density and parallel stresses no longer vanish. So we expect that the obtained formula for the normal force acting on the wedge sides will correctly approximate the corresponding results of more realistic models in the perfectly conducting limit. From (47) we see that the corresponding vacuum forces are attractive for $q > 1$ and repulsive for $q < 1$. In particular, the equilibrium position corresponding to the geometry of a single plate ($q = 1$) is unstable. As regards the part induced by the cylindrical shell, from (48) it follows that $p_{\rm 2cyl} < 0$ and, hence, the corresponding forces are always attractive.

Now, let us discuss the behavior of the boundary-induced part in the VEV of the energy-momentum tensor in the asymptotic region of the parameters. Near the cylindrical shell the main contribution comes from large values of m. Thus, using the uniform asymptotic expansions for the modified Bessel functions for large values of the order, up to the leading order, for the points $a - r \ll a|\sin\phi|, a|\sin(\phi_0 - \phi)|$ we find

$$\langle T_0^0 \rangle_{\rm cyl} \approx -\frac{1}{2}\langle T_2^2 \rangle_{\rm cyl} \approx -\frac{(a-r)^{-3}}{60\pi^2 a}, \quad \langle T_1^1 \rangle_{\rm cyl} \approx \frac{(a-r)^{-2}}{60\pi^2 a^2}. \quad (50)$$

These leading terms are the same as those for a cylindrical shell when the wedge is absent. For the points near the edges ($r = a$, $\phi = 0$ and ϕ_0) the leading terms in the corresponding asymptotic expansions are the same as for the geometry of a wedge with the opening angle $\phi_0 = \pi/2$. The latter are given by (45) with $q = 2$. Near the edge, $r \to 0$, for the components (no summation over i) $\langle T_i^i \rangle_{\rm cyl}$, $i = 0, 3$, the main contribution comes from the mode $m = 0$, and we find

$$\langle T_i^i \rangle_{\rm cyl} \approx \frac{q}{4\pi^2 a^4}\int_0^\infty {\rm d}x\, x^3 \frac{K_0'(x)}{I_0'(x)} = -0.0590\frac{q}{a^4}, \quad i = 0, 3. \quad (51)$$

For the components (no summation over i) $\langle T_i^i \rangle_{\rm cyl}$, $i = 1, 2$, when $q > 1$ the main contribution again comes from the $m = 0$ term, and one has $\langle T_i^i \rangle_{\rm cyl} \approx -\langle T_0^0 \rangle_{\rm cyl}$, $i = 1, 2$. For $q < 1$ the main contribution to the components $\langle T_i^i \rangle_{\rm cyl}$, $i = 1, 2$, comes from the term $m = 1$, and we have (no summation over i)

$$\langle T_i^i\rangle_{\rm cyl} \approx \frac{(-1)^i q\cos(2q\phi)}{2^{2q+1}\pi^2\Gamma^2(q)a^4}\left(\frac{r}{a}\right)^{2(q-1)} \times \int_0^\infty {\rm d}x\, x^{2q+1}\left[\frac{K_q(x)}{I_q(x)}-\frac{K_q'(x)}{I_q'(x)}\right], \quad i=1,2. \tag{52}$$

In this case the radial and azimuthal stresses induced by the cylindrical shell diverge on the edge $r=0$. In the case $q=1$ the sum of the contributions of the terms with $m=0$ and $m=1$ given by (51) and (52) should be taken. For the off-diagonal component the main contribution comes from the $m=1$ mode, with the leading term

$$\langle T_2^1\rangle_{\rm cyl} \approx \frac{q\sin(2q\phi)}{2^{2q+1}\pi^2\Gamma^2(q)a^3}\left(\frac{r}{a}\right)^{2q-1} \times \int_0^\infty {\rm d}x\, x^{2q+1}\left[\frac{K_q(x)}{I_q(x)}-\frac{K_q'(x)}{I_q'(x)}\right], \tag{53}$$

and this component vanishes on the edge for $q>1/2$. Similar to the case of the field square, (51)–(53) also describe the asymptotic behavior for the VEV of the energy-momentum tensor in the limit of large values for the cylindrical shell radius when r is fixed.

In the limit $q \gg 1$, the contribution of the modes with $m \geqslant 1$ is suppressed by the factor $\exp[-2qm\ln(a/r)]$ and the main contribution comes from the $m=0$ mode. The leading terms are given by (no summation over i)

$$\langle T_i^i\rangle_{\rm cyl} \approx \frac{q}{4\pi^2a^4}\int_0^\infty {\rm d}x\, x^3\frac{K_0'(x)}{I_0'(x)}I_0^2(xr/a), \quad i=0,3, \tag{54}$$

$$\langle T_i^i\rangle_{\rm cyl} \approx -\frac{q}{4\pi^2a^4}\int_0^\infty {\rm d}x\, x^3\frac{K_0'(x)}{I_0'(x)} \times \left[I_0^2(xr/a)+(-1)^i I_1^2(xr/a)\right], \quad i=1,2. \tag{55}$$

Though in this limit the vacuum densities are large, due to the factor $1/q$ in the spatial volume, the corresponding global quantities tend to a finite value. In particular, as follows from (55), in the limit under consideration one has $\langle T_i^i\rangle_{\rm cyl}>0$. Note that in the same limit the parts corresponding to the wedge without the cylindrical shell behave as q^4 and, hence, for points not too close to the shell these parts dominate in the VEVs.

In Figs. 2–5 we have plotted the parts in the VEVs of the energy-momentum tensor induced by the cylindrical shell, $a^4\langle T_i^k\rangle_{\rm cyl}$, as functions of $x=(r/a)\cos\phi$ and $y=(r/a)\sin\phi$, for a wedge with the opening angle $\phi_0=\pi/2$.

In Fig. 6 we have presented the dependence of the effective azimuthal pressure induced by the cylindrical shell on the wedge sides, $a^4 p_{2\rm cyl}$, as a function of r/a for different values of the parameter q.

There are several special cases of interest for the geometry of boundaries we have considered. The case $\phi_0=\pi$ corresponds to the semi-circular cylinder. The Casimir energy for the corresponding interior region is evaluated

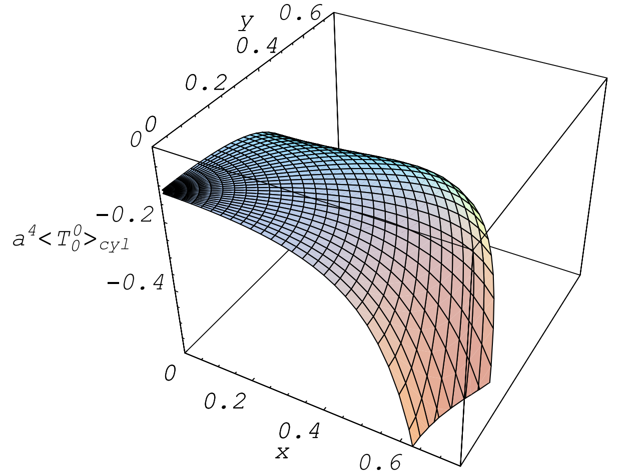

Fig. 2. The part in the VEV of the energy density, $a^4\langle T_0^0\rangle_{\rm cyl}$, induced by the cylindrical boundary as a function of $x=(r/a)\cos\phi$ and $y=(r/a)\sin\phi$ for a wedge with $\phi_0=\pi/2$

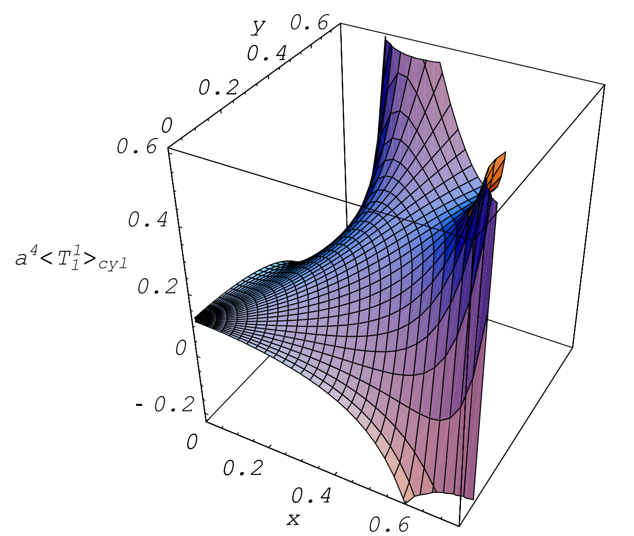

Fig. 3. The part in the VEV of the radial stress, $a^4\langle T_1^1\rangle_{\rm cyl}$, induced by the cylindrical boundary as a function of $x=(r/a)\cos\phi$ and $y=(r/a)\sin\phi$ for a wedge with $\phi_0=\pi/2$

in [58] by using the zeta function technique. The case $\phi_0=2\pi$ corresponds to the geometry of a cylindrical shell with a coaxial half-plane. Finally, the limit $\phi_0\to 0$, $r,a\to\infty$, assuming that $a-r$ and $a\phi_0\equiv b$ are fixed, corresponds to the geometry of two parallel plates separated by a distance b, perpendicularly intersected by the third plate. In the latter case it is convenient to introduce rectangular coordinates $(x'^1,x'^2,x'^3)=(x,y,z)$ with the relations $x=a-r$ and $y=r\phi$. We will denote the components of the tensors in these coordinates by primes. The corresponding vacuum energy-momentum tensor is presented in the form

$$\langle 0|T_k'^i|0\rangle = \langle T_k'^i\rangle^{(0)} + \langle T_k'^i\rangle^{(1)}, \tag{56}$$

where $\langle T_k'^i\rangle^{(0)}$ is the vacuum expectation value in the region between two parallel plates located at $y=0$ and $y=a$,

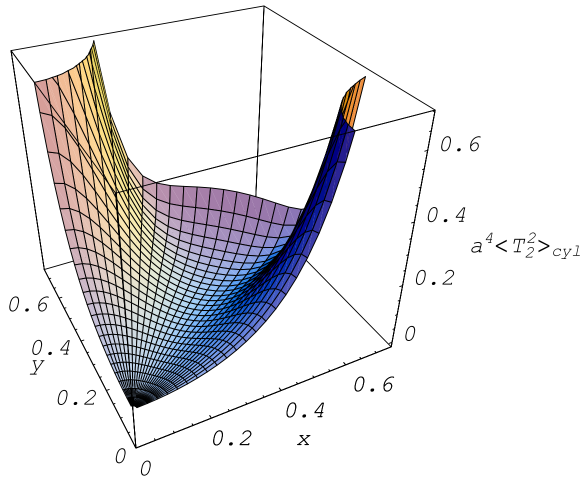

Fig. 4. The part in the VEV of the azimuthal stress, $a^4\langle T_2^2\rangle_{\rm cyl}$, induced by the cylindrical boundary as a function of $x = (r/a)\cos\phi$ and $y = (r/a)\sin\phi$ for a wedge with $\phi_0 = \pi/2$

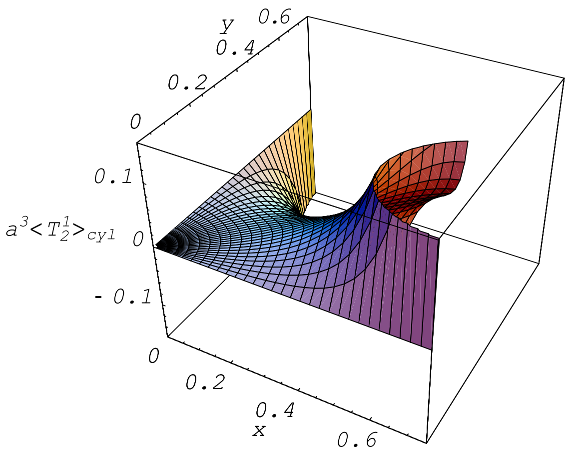

Fig. 5. The part in the VEV of the off-diagonal component, $a^4\langle T_1^2\rangle_{\rm cyl}$, induced by the cylindrical boundary as a function of $x = (r/a)\cos\phi$ and $y = (r/a)\sin\phi$ for a wedge with $\phi_0 = \pi/2$

and $\langle T_k^{\prime i}\rangle^{(1)}$ is induced by the intersecting plate at $x = 0$. The latter is related to the quantities investigated above by

$$\left\langle T_i^{\prime i}\right\rangle^{(1)} = \lim \left\langle T_i^i\right\rangle_{\rm cyl}\,, \quad \left\langle T_2^{\prime 1}\right\rangle^{(1)} = -\lim \frac{1}{a}\left\langle T_2^1\right\rangle_{\rm cyl}\,, \tag{57}$$

with lim corresponding to the limit $a \to \infty$, $\phi_0 \to 0$ for fixed $a - r$ and $a\phi_0$. Taking this limit in the term with $m = 0$ of (38) we replace the modified Bessel functions by the leading terms of the corresponding asymptotic formulae for large values of the argument and the integral is taken elementary. For the terms with $m \neq 0$ in (38) and (39) we note that in the limit under consideration one has $q = \pi/\phi_0 \to \infty$, and the order of the modified Bessel functions tends to infinity. Introducing the new integration variable $x \to qmx$, we can replace these functions by their uniform asymptotic expansions for large values of the order. After these replacements the integration and the further summation over m are done by using the formulae from [74].

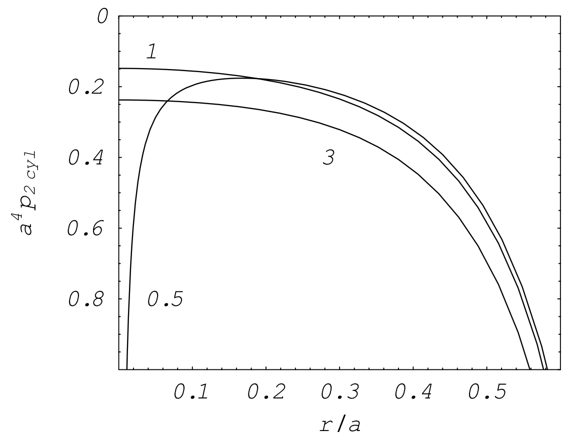

Fig. 6. The effective azimuthal pressure induced by the cylindrical shell on the wedge sides, $a^4 p_{2\rm cyl}$, as a function of r/a. The numbers near the curves correspond to the values of the parameter q

4 Vacuum densities in the exterior region

In this section we consider the VEVs for the field square and the energy-momentum tensor in the region outside the cylindrical boundary (region II in Fig. 1). The corresponding eigenfunctions for the vector potential are obtained from (2) by the replacement

$$\begin{aligned} J_{qm}(\gamma r) &\to g_{qm}^{(\lambda)}(\gamma a, \gamma r) \\ &= J_{qm}(\gamma r) Y_{qm}^{(\lambda)}(\gamma a) - Y_{qm}(\gamma r) J_{qm}^{(\lambda)}(\gamma a)\,, \end{aligned} \tag{58}$$

where, as before, $\lambda = 0, 1$ correspond to the waves of the electric and magnetic types, respectively. Now, the eigenvalues for γ are continuous and in the normalization condition (4) the corresponding part on the right is presented by the delta function. As the normalization integral diverges for $\gamma' = \gamma$, the main contribution to the integral comes from large values of r, and we can replace the cylindrical functions with the argument γr by their asymptotic expressions for large values of the argument. In this way it can be seen that the normalization coefficient in the exterior region is determined by

$$\beta_\alpha^{-2} = \frac{8\pi}{q}\delta_m \gamma\omega \left[J_{qm}^{(\lambda)2}(\gamma a) + Y_{qm}^{(\lambda)2}(\gamma a)\right]. \tag{59}$$

Substituting the eigenfunctions into the corresponding mode-sum formula, for the VEV of the field square one finds

$$\begin{aligned} \langle 0|F^2|0\rangle = \frac{2q}{\pi}\sum_{m=0}^{\infty}{}' \int_{-\infty}^{+\infty} {\rm d}k \int_0^\infty {\rm d}\gamma \sum_{\lambda=0,1} \frac{\gamma^3}{\sqrt{k^2+\gamma^2}} \\ \times \frac{g^{(\eta_{F\lambda})}\left[\varPhi_{qm}^{(\lambda)}(\phi), g_{qm}^{(\lambda)}(\gamma a, \gamma r)\right]}{J_{qm}^{(\lambda)2}(\gamma a) + Y_{qm}^{(\lambda)2}(\gamma a)}, \end{aligned} \tag{60}$$

where the functions $g^{(\eta_{F\lambda})}[\varPhi_{qm}^{(\lambda)}(\phi), g_{qm}^{(\lambda)}(\gamma a, \gamma r)]$ are defined by (9) and (10) with the function $f(x) = g_{qm}^{(\lambda)}(\gamma a, x)$.

To extract from this VEV the part induced by the cylindrical shell, we subtract from the right-hand side the corresponding expression for the wedge without the cylindrical boundary. The latter is given by (14). The corresponding difference can be further evaluated by using the identity

$$\frac{g^{(\eta_F\lambda)}\left[\Phi_{qm}^{(\lambda)}(\phi), g_{qm}^{(\lambda)}(\gamma a, \gamma r)\right]}{J_{qm}^{(\lambda)2}(\gamma a)+Y_{qm}^{(\lambda)2}(\gamma a)} = g^{(\eta_F\lambda)}\left[\Phi_{qm}^{(\lambda)}(\phi), J_{qm}(\gamma r)\right] - \frac{1}{2}\sum_{s=1}^{2}\frac{J_{qm}^{(\lambda)}(\gamma a)}{H_{qm}^{(s)(\lambda)}(\gamma a)} g^{(\eta_F\lambda)}\left[\Phi_{qm}^{(\lambda)}(\phi), H_{qm}^{(s)}(\gamma r)\right], \quad (61)$$

where $H_{qm}^{(1,2)}(z)$ are the Hankel functions. In order to transform the integral over γ with the last term on the right of (61), in the complex γ-plane we rotate the integration contour by the angle $\pi/2$ for the term with $s=1$ and by the angle $-\pi/2$ for the term with $s=2$. Due to the well-known properties of the Hankel functions the integrals over the corresponding parts of the circles of large radius in the upper and lower half-planes vanish. After introducing the modified Bessel functions and integrating over k with the help of (26), we can write the VEVs of the field square in the form (13), where the part induced by the cylindrical shell is given by

$$\langle F^2\rangle_{\rm cyl} = \frac{2q}{\pi}\sum_{m=0}^{\infty}{}'\sum_{\lambda=0,1}\int_0^{\infty}{\rm d}x\, x^3 \frac{I_{qm}^{(\lambda)}(xa)}{K_{qm}^{(\lambda)}(xa)} G^{(\eta_F\lambda)} \times\left[\Phi_{qm}^{(\lambda)}(\phi), K_{qm}(xr)\right]. \quad (62)$$

In this formula the functions $G^{(\eta_F\lambda)}[\Phi(\phi), f(x)]$ are defined by (28) and (29). Comparing this result with (27), we see that the expressions for the shell-induced parts in the interior and exterior regions are related by the interchange $I_{qm} \rightleftarrows K_{qm}$. The VEV (62) diverges on the cylindrical shell with the leading term being the same as that for the interior region. At large distances from the cylindrical shell we introduce a new integration variable $y = xr$ and expand the integrand over a/r. For $q>1$ the main contribution comes from the lowest mode $m=0$ and up to the leading order we have

$$\langle E^2\rangle_{\rm cyl} \approx \frac{4q}{5\pi r^4}\left(\frac{a}{r}\right)^2, \quad \langle B^2\rangle_{\rm cyl} \approx -\frac{28q}{15\pi r^4}\left(\frac{a}{r}\right)^2. \quad (63)$$

For $q<1$ the dominant contribution into the VEVs at large distances is due to the mode $m=1$ with the leading term

$$\langle F^2\rangle_{\rm cyl} \approx -\frac{4q^2(q+1)}{\pi r^4}\left(\frac{a}{r}\right)^{2q}\left[\frac{\cos(2q\phi)}{2q+3} + (-1)^{\eta_{F1}}\frac{q+1}{2q+1}\right]. \quad (64)$$

For the case $q=1$ the contributions of the modes $m=0$ and $m=1$ are of the same order and the corresponding leading terms are obtained by summing these contributions. The latter are given by the right-hand sides of (63) and (64). As we see, at large distances the part induced by the cylindrical shell is suppressed with respect to the part corresponding to the wedge without the shell by the factor $(a/r)^{2\beta}$ with $\beta = \min(1, q)$.

Now we turn to the VEVs of the energy-momentum tensor in the exterior region. Substituting the eigenfunctions into the corresponding mode-sum formula, one finds (no summation over i)

$$\langle 0|T_i^i|0\rangle = \frac{q}{4\pi^2}\sum_{m=0}^{\infty}{}'\int_{-\infty}^{+\infty}{\rm d}k\int_0^{\infty}{\rm d}\gamma \times\sum_{\lambda=0,1}\frac{\gamma^3}{\sqrt{k^2+\gamma^2}}\frac{f^{(i)}\left[\Phi_{qm}^{(\lambda)}(\phi), g_{qm}^{(\lambda)}(\gamma a, \gamma r)\right]}{J_{qm}^{(\lambda)2}(\gamma a)+Y_{qm}^{(\lambda)2}(\gamma a)}, \quad (65)$$

$$\langle 0|T_2^1|0\rangle = -\frac{q}{8\pi^2}\frac{\partial}{\partial r}\sum_{m=0}^{\infty}{}' m\sin(2qm\phi)\int_{-\infty}^{+\infty}{\rm d}k\int_0^{\infty}{\rm d}\gamma \times\sum_{\lambda=0,1}(-1)^{\lambda}\frac{\gamma g_{qm}^{(\lambda)2}(\gamma a, \gamma r)}{\sqrt{k^2+\gamma^2}}. \quad (66)$$

Subtracting from these VEVs the corresponding expression for the wedge without the cylindrical boundary, analogously to the case of the field square, it can be seen that the VEVs are presented in the form (37), with the parts induced by the cylindrical shell given by (no summation over i)

$$\langle T_i^i\rangle_{\rm cyl} = \frac{q}{2\pi^2}\sum_{m=0}^{\infty}{}'\sum_{\lambda=0,1}\int_0^{\infty}{\rm d}x x^3\frac{I_{qm}^{(\lambda)}(xa)}{K_{qm}^{(\lambda)}(xa)} \times F^{(i)}\left[\Phi_{qm}^{(\lambda)}(\phi), K_{qm}(xr)\right], \quad (67)$$

$$\langle T_2^1\rangle_{\rm cyl} = \frac{q^2}{4\pi^2}\frac{\partial}{\partial r}\sum_{m=0}^{\infty}{}' m\sin(2qm\phi)\sum_{\lambda=0,1}(-1)^{\lambda} \times\int_0^{\infty}{\rm d}x x\frac{I_{qm}^{(\lambda)}(xa)}{K_{qm}^{(\lambda)}(xa)}K_{qm}^2(xr). \quad (68)$$

Here the functions $F^{(i)}[\Phi(\phi), f(y)]$ are defined by (40) and (41). By using the inequality given in the paragraph after (43), we can show that the vacuum energy density induced by the cylindrical shell in the exterior region is positive.

In a way similar to that for the interior region, the force acting on the wedge sides is presented in the form of the sum (46), where the part corresponding to the wedge without a cylindrical shell is determined by (47) and for the part due to the presence of the cylindrical shell we have

$$p_{2\rm cyl} = -\langle T_2^2\rangle_{\rm cyl}|_{\phi=0,\phi_0} = -\frac{q}{\pi^2}\sum_{m=0}^{\infty}{}'\sum_{\lambda=0,1}\int_0^{\infty}{\rm d}x x^3\frac{I_{qm}^{(\lambda)}(xa)}{K_{qm}^{(\lambda)}(xa)}F_{qm}^{(\lambda)}\left[K_{qm}(xr)\right]. \quad (69)$$

In this formula, the function $F_{\nu}^{(\lambda)}[f(y)]$ is defined by (49) and the corresponding forces are always attractive.

The leading divergence in the boundary-induced part (67) on the cylindrical surface is given by the same formulae as for the interior region. For large distances from the shell and for $q > 1$ the main contribution to the VEVs of the diagonal components comes from the $m = 0, \lambda = 1$ term and one has (no summation over i)

$$\langle T_i^i \rangle_{\rm cyl} \approx -\frac{qc_i}{15\pi^2 r^4}\left(\frac{a}{r}\right)^2, \quad c_0 = c_3 = 2, \quad c_1 = 1, \quad c_2 = -5. \tag{70}$$

In the case $q < 1$ the main contribution to the VEVs of the diagonal components at large distances from the cylindrical shell comes from the $m = 1$ mode. The leading terms in the corresponding asymptotic expansions are given by

$$\langle T_i^i \rangle_{\rm cyl} \approx -q^2(q+1)c_i(q)\frac{\cos(2q\phi)}{\pi^2 r^4}\left(\frac{a}{r}\right)^{2q}, \tag{71}$$

with the notation

$$c_0(q) = c_3(q) = \frac{1}{2q+3}, \quad c_1(q) = \frac{2q^2+q+1}{(2q+1)(2q+3)}, \quad c_2(q) = -\frac{q+1}{2q+1}. \tag{72}$$

In the case $q = 1$ the asymptotic terms are determined by the sum of the contributions coming from the modes $m = 0$ and $m = 1$. The latter are given by (70) and (71). For the off-diagonal component, for all values q the main contribution at large distances comes from the $m = 1$ mode, with the leading term

$$\langle T_2^1 \rangle_{\rm cyl} \approx -\frac{q^3(q+1)}{2q+1}\frac{\sin(2q\phi)}{\pi^2 r^3}\left(\frac{a}{r}\right)^{2q}. \tag{73}$$

For large values of q, $q \gg 1$, the contribution of the terms with $m > 0$ is suppressed by the factor $\exp[-2qm\ln(r/a)]$ and the main contribution comes form the $m = 0$ term with the behavior $\langle F^2 \rangle_{\rm cyl} \propto q$ and $\langle T_i^k \rangle_{\rm cyl} \propto q$. In Fig. 7 we have plotted the dependence of the effective azimuthal pressure induced by the cylindrical shell on the wedge sides, $a^4 p_{2\rm cyl}$, as a function of r/a for $q = 1$.

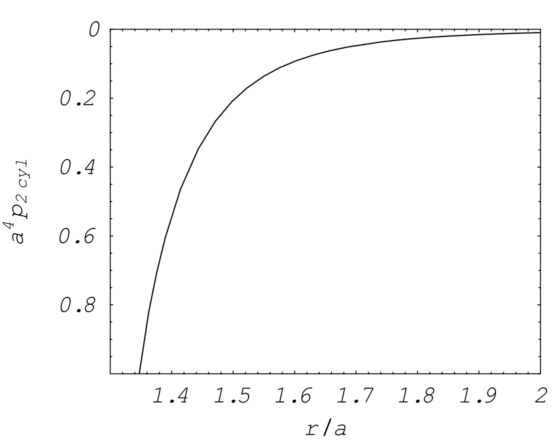

Fig. 7. The effective azimuthal pressure induced by the cylindrical shell on the wedge sides, $a^4 p_{2\rm cyl}$, as a function of r/a in the exterior region for $q = 1$. The curves for the values $q = 0.5, 3$ are close to the plotted one

In order to investigate the influence of the conducting boundaries on the properties of the electromagnetic vacuum we have considered a model where the physical interactions are replaced by the imposition of boundary conditions on the electric and magnetic fields for all modes. Of course, this is an idealization, as real physical materials cannot constrain all the modes of a fluctuating quantum field [10–14, 53]. In general, the physical quantities in the problems with boundary conditions can be classified into two main groups (see also the last paper in [10–14]). The first group includes quantities that do not contain surface divergences. For these quantities the renormalization procedure is the same as in quantum field theory without boundaries and they can be evaluated by boundary condition calculations. The contribution of the higher modes to the boundary induced effects in these quantities is suppressed by the parameters already present in the idealized model. Examples of such quantities are the energy density and the vacuum stresses at the points away from the boundary and the interaction forces between disjoint bodies. For instance, if we consider the energy density at the point having a distance x from a perfectly conducting boundary, the main contribution comes from the frequencies $\lesssim 1/x$. The formulae obtained for this type of quantities on the basis of the idealized model are valid for real conductors up to distances x for which $1/x \ll \omega_0$, with ω_0 being the characteristic frequency, such that for $\omega > \omega_0$ the conditions for perfect conductivity fail. In particular, the results for the interaction forces between disjoint bodies of various geometries obtained by boundary condition calculations are confirmed by recent experiments with high accuracy. An interesting topic for further research in this direction is to take into account the effects of finite conductivity for constraining boundaries.

For the quantities from the second group, such as the energy density on the boundary and the total vacuum energy, the contribution of the arbitrary higher modes is dominant, and they contain divergences that cannot be eliminated by the standard renormalization procedure of quantum field theory without boundaries. Of course, the model in which the physical interaction is replaced by the imposition of boundary conditions on the field for all modes is an idealization. The appearance of divergences in the process of the evaluation of physical quantities of the second type indicates that a more realistic physical model should be employed for their evaluation. In the literature on the Casimir effect different field-theoretical approaches have been discussed to extract the finite parts from the diverging quantities. However, in the physical interpretation of these results it should be taken into account that these terms are only a part of the full expression of the physical quantity and the terms that are divergent in the idealized model can be physically essential and their evaluation needs a more realistic model. It seems plausible that such effects as surface roughness, or the microstructure of the boundary on small scales can introduce a physical cutoff needed to produce finite values for surface quantities (see, for instance, [75–78] and references therein for models of this kind). Another possibility, proposed in [10–14], is to replace a boundary condition by a renormalizable coupling between the fluctuating field and a non-dynamical smooth background field representing the material (for the evaluation of the vacuum energy in smooth background fields see also [6–9]). In this model the standard renormalization procedure of quantum field theory without boundaries provides the finite result for the quantities that are divergent in the boundary condition limit. An alternative mechanism for introducing a cutoff that removes the singular behavior on the boundaries is to allow the position of the boundary to undergo quantum fluctuations [79]. Such fluctuations smear out the contribution of the high frequency modes without the need to introduce an explicit high frequency cutoff. The main subject of the present paper is the investigation of the VEVs for the field square and the energy-momentum tensor of the electromagnetic field at the points away from the boundaries and the normal vacuum forces acting on the wedge sides. These quantities fall into the first group. They do not contain surface divergences and are completely determined within the framework of standard QED. We expect that similar results would be obtained in the model in which instead of externally imposed boundary condition the fluctuating field is coupled to a smooth background potential that implements the boundary condition in a certain limit [10–14].

5 Conclusion

In this paper we have investigated the polarization of the electromagnetic vacuum by a wedge with a coaxial cylindrical boundary, assuming that all boundaries are perfectly conducting. Both regions inside and outside of the cylindrical shell (regions I and II in Fig. 1) are considered. In Sect. 2 we have evaluated the VEVs of the field square in the interior region. The corresponding mode sums contain series over the zeros of the Bessel function for TM modes and its derivative for TE modes. For the summation of these series we used a variant of the generalized Abel–Plana formula. The latter enables us to extract from the VEVs the parts corresponding to the geometry of a wedge without a cylindrical shell and to present the parts induced by the shell in terms of integrals that are exponentially convergent for points away from the boundaries. For the wedge without the cylindrical shell the VEVs of the field square are presented in the form (23). The first term on the right of this formula corresponds to the VEVs for the geometry of a cosmic string with the angle deficit $2\pi - \phi_0$. The angle-dependent parts in the VEVs of the electric and magnetic fields have opposite signs and are cancelled in the evaluation of the vacuum energy density. The parts induced by the cylindrical shell are presented in the form (27). We have discussed this general formula in various asymptotic regions of the parameters including the points near the edges and near the shell. In Sect. 3 we consider the VEV of the energy-momentum tensor in the region inside the shell. As for the field square, the application of the Abel–Plana formula allows us to present this VEV in the form of the sum of purely wedge and shell-induced parts, (37). For the geometry of a wedge without the cylindrical boundary the vacuum energy-momentum tensor does not depend on the angle ϕ and is the same as in the geometry of the cosmic string; it is given by (45). The corresponding vacuum forces acting on the wedge sides are attractive for $\phi_0 < \pi$ and repulsive for $\phi_0 > \pi$. In particular, the equilibrium position corresponding to the geometry of a single plate ($\phi_0 = \pi$) is unstable. For the region inside the shell the part in the VEV of the energy-momentum tensor induced by the presence of the cylindrical shell is non-diagonal and the corresponding components are given by (38) and (39). The vacuum energy density induced by the cylindrical shell in the interior region is negative. We have investigated the vacuum densities induced by the cylindrical shell in various asymptotic regions of the parameters. For points near the cylindrical shell the leading terms in the asymptotic expansions over the distance from the shell are given by (50). These terms are the same as those for a cylindrical shell when the wedge is absent. For a wedge with $\phi_0 < \pi$ the part in the vacuum energy-momentum tensor induced by the shell is finite on the edge $r = 0$. For $\phi_0 > \pi$ the shell-induced parts in the energy density and the axial stress remain finite, whereas the radial and azimuthal stresses diverge as $r^{2(\pi/\phi_0 - 1)}$. The corresponding off-diagonal component behaves like $r^{2\pi/\phi_0 - 1}$ for all values ϕ_0. For the points near the edges ($r = a$, $\phi = 0$ and ϕ_0) the leading terms in the corresponding asymptotic expansions are the same as for the geometry of a wedge with the opening angle $\phi_0 = \pi/2$. In the limit of small opening angles, $\phi_0 \ll \pi$, the shell-induced parts behave like $1/\phi_0$. In the same limit the parts corresponding to the wedge without the shell behave as $1/\phi_0^4$, and for points not too close to the shell these parts dominate in the VEV of the energy-momentum tensor. The presence of the shell leads to additional forces acting on the wedge sides. The corresponding effective azimuthal pressure is given by (48) and these forces are always attractive.

The VEVs of the field square and the energy-momentum tensor in the region outside the cylindrical shell are investigated in Sect. 4. As in the case of the interior region, these VEVs are presented as sums of the parts corresponding to the wedge without the cylindrical shell and the parts induced by the shell. The latter are given by (62) for the field square and by (67) and (66) for the components of the energy-momentum tensor. In the exterior region the vacuum energy density induced by the cylindrical shell is always positive. Additional forces acting on the wedge sides due to the presence of the shell are given by (69). As in the case of the interior region these forces are attractive. For large values of the parameter q, the contribution into the parts induced by the cylindrical shell coming from the modes with $m \neq 0$ is exponentially suppressed, whereas the contribution of the lowest mode, $m = 0$, is proportional to q. Though in this limit the vacuum densities are large, due to the factor $1/q$ in the spatial volume element, the corresponding global quantities tend to finite limiting values.

Acknowledgements. The work was supported by the Armenian Ministry of Education and Science Grant No. 0124.

References

1. H.B.G. Casimir, Proc. K Ned. Akad. Wet. **51**, 793 (1948)
2. V.M. Mostepanenko, N.N. Trunov, The Casimir Effect and Its Applications (Oxford University Press, Oxford, 1997)
3. M. Bordag, U. Mohidden, V.M. Mostepanenko, Phys. Rep. **353**, 1 (2001)
4. K.A. Milton, The Casimir Effect: Physical Manifestation of Zero-Point Energy (World Scientific, Singapore, 2002)
5. V.V. Nesterenko, G. Lambiase, G. Scarpetta, Riv. Nuovo Cimento **27**, 1 (2004)
6. M. Bordag, J. Phys. A **28**, 755 (1995)
7. M. Bordag, G. Lindig, J. Phys. A **29**, 4481 (1996)
8. M. Bordag, K. Kirsten, Phys. Rev. D **53**, 5753 (1996)
9. M. Bordag, K. Kirsten, Phys. Rev. D **60**, 105 019 (1999)
10. N. Graham, R.L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, H. Weigel, Nucl. Phys. B **645**, 49 (2002)
11. N. Graham, R.L. Jaffe, H. Weigel, Int. J. Mod. Phys. A **17**, 846 (2002)
12. N. Graham, R.L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, H. Weigel, Phys. Lett. B **572**, 196 (2003)
13. N. Graham, R.L. Jaffe, V. Khemani, M. Quandt, O. Schröder, H. Weigel, Nucl. Phys. B **677**, 379 (2004)
14. R.L. Jaffe, A. Scardicchio, JHEP **06**, 006 (2006)
15. M. Schaden, L. Spruch, Phys. Rev. A **58**, 935 (1998)
16. M. Schaden, L. Spruch, Phys. Rev. Lett. **84**, 459 (2000)
17. R. Golestanian, M. Kardar, Phys. Rev. A **58**, 1713 (1998)
18. T. Emig, R. Buscher, Nucl. Phys. B **696**, 468 (2004)
19. H. Gies, K. Langfeld, L. Moyaerts, JHEP **06**, 018 (2003)
20. R.L. Jaffe, A. Scardicchio, Phys. Rev. Lett. **92**, 070 402 (2004)
21. A. Scardicchio, R.L. Jaffe, Nucl. Phys. B **704**, 552 (2005)
22. A. Edery, J. Phys. A **39**, 685 (2006)
23. A. Bulgac, P. Magierski, A. Wirzba, Phys. Rev. D **73**, 025 007 (2006)
24. T. Emig, R.L. Jaffe, M. Kardar, A. Scardicchio, Phys. Rev. Lett. **96**, 080 403 (2006)
25. M. Bordag, Phys. Rev. D **73**, 125 018 (2006)
26. H. Gies, K. Klingmuller, Phys. Rev. Lett. **96**, 220 401 (2006)
27. P.M. Fishbane, S.G. Gasiorowich, P. Kauss, Phys. Rev. D **36**, 251 (1987)
28. P.M. Fishbane, S.G. Gasiorowich, P. Kauss, Phys. Rev. D **37**, 2623 (1988)
29. B.M. Barbashov, V.V. Nesterenko, Introduction to the Relativistic String Theory (World Scientific, Singapore, 1990)
30. J. Ambjørn, S. Wolfram, Ann. Phys. **147**, 1 (1983)
31. L.L. De Raad Jr., K.A. Milton, Ann. Phys. **136**, 229 (1981)
32. K.A. Milton, A.V. Nesterenko, V.V. Nesterenko, Phys. Rev. D **59**, 105 009 (1999)
33. P. Gosdzinsky, A. Romeo, Phys. Lett. B **441**, 265 (1998)
34. G. Lambiase, V.V. Nesterenko, M. Bordag, J. Math. Phys. **40**, 6254 (1999)
35. I. Cavero-Peláez, K.A. Milton, J. Phys. A **39**, 6225 (2006)
36. A. Romeo, K.A. Milton, J. Phys. A **39**, 6225 (2006)
37. I. Brevik, A. Romeo, Phys. Scripta **76**, 48 (2007)
38. A.A. Saharian, Izv. Akad. Nauk. Arm. SSR. Fiz. **23**, 130 (1988) [Sov. J. Contemp. Phys. **23**, 14 (1988)]
39. A.A. Saharian, Dokl. Akad. Naur. Arm. SSR **86**, 112 (1988) (Reports NAS RA, in Russian)
40. A.A. Saharian, The generalized Abel–Plana formula. Applications to Bessel functions and Casimir effect, Report No. IC/2000/14 (hep-th/0002239)
41. F.D. Mazzitelli, M.J. Sanchez, N.N. Scoccola, J. von Stecher, Phys. Rev. A **67**, 013 807 (2002)
42. D.A.R. Dalvit, F.C. Lombardo, F.D. Mazzitelli, R. Onofrio, Europhys. Lett. **67**, 517 (2004)
43. F.D. Mazzitelli, in Quantum Field Theory under the Influence of External Conditions, ed. by K.A. Milton (Rinton Press, Princeton NJ, 2004)
44. A. Romeo, A.A. Saharian, Phys. Rev. D **63**, 105 019 (2001)
45. A.A. Saharian, A.S. Tarloyan, J. Phys. A **39**, 13 371 (2006)
46. F.D. Mazzitelli, D.A.R. Dalvit, F.C. Lombardo, New J. Phys. **8**, 240 (2006)
47. A.R. Kitson, A. Romeo, Phys. Rev. D **74**, 085 024 (2006)
48. V.N. Marachevsky, Phys. Rev. D **75**, 085 019 (2007)
49. J.S. Apps, J.S. Dowker, Class. Quantum Grav. **15**, 1121 (1998)
50. J.S. Dowker, Divergences in the Casimir energy, hep-th/0006138
51. V.V. Nesterenko, I.G. Pirozhenko, J. Dittrich, Class. Quantum Grav. **20**, 431 (2003)
52. J.S. Dowker, G. Kennedy, J. Phys. A **11**, 895 (1978)
53. D. Deutsch, P. Candelas, Phys. Rev. D **20**, 3063 (1979)
54. I. Brevik, M. Lygren, Ann. Phys. **251**, 157 (1996)
55. I. Brevik, M. Lygren, V.N. Marachevsky, Ann. Phys. **267**, 134 (1998)
56. I. Brevik, K. Pettersen, Ann. Phys. **291**, 267 (2001)
57. V.V. Nesterenko, G. Lambiase, G. Scarpetta, Ann. Phys. **298**, 403 (2002)
58. V.V. Nesterenko, G. Lambiase, G. Scarpetta, J. Math. Phys. **42**, 1974 (2001)
59. A.H. Rezaeian, A.A. Saharian, Class. Quantum Grav. **19**, 3625 (2002)
60. A.A. Saharian, A.S. Tarloyan, J. Phys. A **38**, 8763 (2005)
61. A.A. Saharian, Izv. Akad. Nauk. Arm. SSR Mat. **22**, 166 (1987) (Sov. J. Contemp. Math. Anal. **22**, 70 (1987))
62. A.A. Saharian, Generalized Abel–Plana formula as a renormalization tool in quantum field theory with boundaries. Proceedings of the Fifth International Conference on Mathematical Methods in Physics, 24–28 April, 2006, Rio de Janeiro, Brazil, PoS(IC2006)019, hep-th/0609093
63. E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, A.S. Tarloyan, Phys. Rev. D **74**, 025 017 (2006)
64. E.R. Bezerra de Mello, V.B. Bezerra, A.A. Saharian, Phys. Lett. B **645**, 245 (2007)
65. J.D. Jackson, Classical Electrodynamics (John Wiley & Sons, New York, 1998)
66. A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev, Integrals and Series (Gordon and Breach, New York, 1986), Vol. 2
67. P.C.W. Davies, V. Sahni, Class. Quantum Grav. **5**, 1 (1988)
68. M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions (Dover, New York, 1972)
69. V.P. Frolov, E.M. Serebriany, Phys. Rev. D **35**, 3779 (1987)
70. J.S. Dowker, Phys. Rev. D **36**, 3742 (1987)
71. A.D. Helfer, A.S.I.D. Lang, J. Phys. A **32**, 1937 (1999);
72. V. Sopova, L.H. Ford, Phys. Rev. D **66**, 045 026 (2002)
73. G. Barton, J. Phys. A **38**, 3021 (2005)
74. A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series (Gordon and Breach, New York, 1986), Vol. 1
75. V.N. Marachevsky, Mod. Phys. Lett. A **16**, 1007 (2001)
76. G. Barton, J. Phys. A **34**, 4083 (2001)
77. G. Barton, J. Phys. A **37**, 1011 (2004)
78. G. Barton, J. Phys. A **38**, 2997 (2005)
79. L.H. Ford, N.F. Svaiter, Phys. Rev. D **58**, 065 007 (1998)